

Energy definition for quadratic curvature gravities

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A conserved current for generic quadratic curvature gravitational models is defined and it is shown that at the linearized level it corresponds to the Deser-Tekin charges. An explicit expression for the charge for new massive gravity in three dimensions is given. Some implications of the linearized equations are discussed.

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I. INTRODUCTION

Defining a local energy density for a given gravitating system is still a matter of discussion, however, it is possible to define a total energy in the general theory of relativity [1]. Closely following the definition of conserved charges in the general theory of relativity [2], a new definition of energy for the quadratic curvature (QC) models has been proposed in [3] which will be called Deser-Tekin (DT) charges (see, also the previous definitions [4]). In particular, the conserved charge corresponding to the DT energy for asymptotically flat or constant curvature geometries have desirable properties, for example, that it circumvents the zero-energy theorem for conformal gravity in four dimensions [5].

Sparling forms [6] are also used to define energy in the general relativity which are well-suited to a treatment in terms of differential forms relative to an orthonormal coframe [7]. In a direction parallel to the present work, (generalized) Sparling forms for dimensionally-continued Euler forms in arbitrary dimensions were introduced in [8]. The Sparling forms can also be related to the so-called Arnowit-Deser-Misner (ADM) energy [1] relative to a coordinate basis and other energy-momentum pseudo-tensors such as Landau-Lifschitz energy pseudo-tensor as well [9]. They also play an important role in the hamiltonian formulation of General theory of relativity [10] and even in the analysis of gravitational radiation in post-Newtonian approximation for theories with torsion [11]. In the present work, by using the language of the exterior algebra and orthonormal coframe formulation for the QC gravity models, a conserved quantity will be defined and its connection with the DT energy will be discussed.

The rest of the paper is organized as follows. In the following section, metric QC field equations are described in a form that is useful in defining an on-shell conserved current and a total energy. In the subsequent section, the linearized QC equations are studied to further investigate the relation of the energy definition presented with the DT charges. In the last section, an explicit expression for the conserved current in the form of an exact differential form for New massive gravity in three dimensions is presented.

II. QC FIELD EQUATIONS AND DEFINITION OF A NEW CONSERVED CURRENT

The notation and the conventions for the geometrical quantities closely follow those of [12, 13], see also [9]. For the technical details related, in particular, to the QC field equations relative to an orthonormal coframe, the reader is referred to [12].

A convenient starting point is to recall that the identity $*G^\alpha = -\frac{1}{2}\Omega_{\mu\nu} \wedge *\theta^{\alpha\mu\nu}$ for the Einstein form can be used to split it into the sum of two terms as

$$*G^\alpha = d * F^\alpha + *t^\alpha \quad (1)$$

where F^α is known as Sparling 2-form [6] with the expression

$$*F^\alpha = -\frac{1}{2}\omega_{\mu\nu} \wedge *\theta^{\alpha\mu\nu} \quad (2)$$

whereas the pseudo-tensorial energy-momentum 3-form $*t^\alpha$ is explicitly given by

$$*t^\alpha = \frac{1}{2}(\omega_{\mu\nu} \wedge \omega^\alpha{}_\lambda \wedge *\theta^{\mu\nu\lambda} - \omega_{\mu\lambda} \wedge \omega^\lambda{}_\nu \wedge *\theta^{\mu\nu\alpha}). \quad (3)$$

The standard ADM energy expression for an asymptotically flat spacetime follows from the integral of $*F^0$, namely

$$E_{ADM} = \frac{1}{16\pi G} \int_\Sigma *F^0 \quad (4)$$

where Σ is a spacelike hypersurface (see, e.g., [9], p. 118). For the well-known example of Schwarzschild metric, by introducing the isotropic coordinates, the formulae (4), singling out the leading order terms in the metric asymptotically, yields the mass parameter of the metric [9].

As advertised above, it is possible to define conserved current, or equivalently an exact form, for generic QC gravity by splitting the field equations in a way similar to (1), cf. Eqn. (13) below. For the sake of clarity for the arguments and the definitions, mainly pure QC gravity models that follow from the general lagrangian density of the form

$$\mathcal{L}_{qc} = aR^2 *1 + bR^\alpha \wedge *R_\alpha \quad (5)$$

will be considered (a, b are dimensionless coupling constants), confining the discussion to four and subsequently to three dimensions. For the use of first order formalism with the constraints $\omega_{\alpha\beta} + \omega_{\beta\alpha} = 0$ (metric compatibility) and $\Theta^\alpha = 0$ (zero-torsion), one starts with the extended lagrangian

$$\mathcal{L}_e = \mathcal{L}_{qc} + \lambda^\alpha \wedge \Theta_\alpha \quad (6)$$

where λ^α is a vector-valued 2-form imposing the dynamical constraint $\Theta^\alpha = 0$. The metric compatibility constraint can simply be incorporated into the variational procedure by $\delta\omega_{\alpha\beta} + \delta\omega_{\beta\alpha} = 0$. Subsequently, by taking the variational derivative of $\mathcal{L}_e[\theta^\alpha, \omega^\alpha_\beta, \lambda^\alpha]$, one obtains metric equations from the coframe equations expressed in terms of the pseudo-Riemannian quantities only. For the details of the derivation of the QC field equations relative to an orthonormal coframe refer to [12, 13]. It follows that the vacuum equations for the QC lagrangian density (5) then can be written in the form

$$*E^\alpha \equiv D\lambda^\alpha + *T^\alpha = 0 \quad (7)$$

where the vector valued 1-form $E^\alpha = E^\alpha_\beta \theta^\beta$ is defined by (7) and is a symmetric vector-valued 1-form (by invariance of the action under local coframe rotation). As in the Einstein-Hilbert case, it is covariantly constant, $D * E^\alpha = 0$ as a result of diffeomorphism invariance of the QC lagrangian. Moreover, the pure QC terms are contained in $*T^\alpha$ term where $T^\alpha = T^\alpha_\beta \theta^\beta$ is another symmetric vector-valued 1-form. In a generic QC model, fourth order lagrange multiplier term, $D\lambda^\alpha$, and second order terms, $*T^\alpha$, are both expressible in terms of auxiliary antisymmetric-tensor-valued 2-form $X^{\alpha\beta}$. For the QC lagrangian (5) it is explicitly given by

$$X^{\alpha\beta} = \theta^\alpha \wedge (bR^\beta + aR\theta^\beta) - \theta^\beta \wedge (R^\alpha + aR\theta^\alpha). \quad (8)$$

The connection equations can be solved uniquely for lagrange multiplier and consequently, the lagrange multiplier 2-forms λ^β can be found in terms of the auxiliary 2-form as

$$\begin{aligned} \lambda^\beta &= 2i_\alpha \Pi^{\alpha\beta} + \frac{1}{2}\theta^\beta \wedge i_\mu i_\nu \Pi^{\mu\nu} \\ &= 2 * D[bR^\beta + (2a + \frac{1}{2}b)R\theta^\beta] \end{aligned} \quad (9)$$

with $\Pi^{\alpha\beta} = D * X^{\alpha\beta}$ whereas the second term in (7) has the explicit form

$$*T^\alpha = \frac{1}{2}(\Omega^{\mu\nu} \wedge i^\alpha * X_{\mu\nu} - i^\alpha \Omega^{\mu\nu} \wedge * X_{\mu\nu}). \quad (10)$$

Note that the trace of $*T^\alpha$ is $T^\alpha_\alpha * 1 = (4 - n)\mathcal{L}_{qc}$ in n dimensions. The form of the field equation (7), which has been introduced in [12], directly allows one to define a conserved quantity by inspection. In order to do so, first note that the fourth order term can simply be written out as

$$D\lambda^\alpha = d\lambda^\alpha + \omega^\alpha_\beta \wedge \lambda^\beta \quad (11)$$

which in fact separates the terms containing fourth order partial derivatives of the metric components from the lower orders relative to a coordinate basis. This then motivates the definition of a convenient vector-valued 1-form $\tau^\alpha = \tau^\alpha_\beta \theta^\beta$ by

$$*\tau^\alpha \equiv *T^\alpha + \omega^\alpha_\beta \wedge \lambda^\beta. \quad (12)$$

Thus, by making use of the definitions (11) and (12) in the vacuum QC gravity equations, (7) then leads to the compact equations as

$$d\lambda^\alpha + *\tau^\alpha = 0. \quad (13)$$

Consequently, the straightforward splitting renders $*\tau^\alpha$ an exact form and, by the operator identity $d^2 \equiv 0$, it is also a closed form $d*\tau^\alpha = 0$. Note however that $*\tau^\alpha$ is, as in the case of the Sparling form, pseudo-tensorial since the right hand side of (12) depends explicitly on the connection 1-form. On the other hand, for QC gravity, λ^α will always be linear in curvature components. Therefore, the energy-momentum 3-form (12) qualifies to define a conserved current for generic QC models (5). Consequently, it is possible to define the four momentum P^α as

$$P^\alpha(\Sigma) \equiv \int_{\partial\Sigma} \lambda^\alpha = - \int_{\Sigma} *\tau^\alpha \quad (14)$$

where Σ is a space-like hypersurface with boundary $\partial\Sigma$. By definition, $P^\alpha(\Sigma)$ vanishes identically for $\partial\Sigma = \emptyset$. The definition of $*\tau^\alpha$ is independent of matter content in the sense that any matter energy-momentum can simply be included in the right hand side of (12) provided that it is covariantly constant. By inspecting the QC field equations written out relative to a coordinate basis (see, e.g., Eqn. (3) in Ref. [3]), one concludes that it is not straightforward to obtain the splitting (13) relative to a coordinate basis.

III. LINEARIZED QC EQUATIONS

A remarkable property of the conserved current λ^α appears to be related to the linearized form of the vacuum equations (7). In order to establish connection with explicit expression defining energy presented in [3], let us consider the linearization of the field equations (7) by introducing the perturbation $h_{\alpha\beta}$ to the metric $g = \eta_{\alpha\beta}\theta^\alpha \otimes \theta^\beta$ with $\eta_{\alpha\beta} = \text{diag}(-+++)$. In terms of a local orthonormal basis of coframe 1-forms, they can be defined as

$$\theta^\alpha \approx dx^\alpha + h^\alpha{}_\beta dx^\beta. \quad (15)$$

The indices of the symmetric $h_{\alpha\beta}$ are raised and lowered by the flat Minkowski metric η in the linear approximation. The relevant tensorial quantities which are linear in $h_{\alpha\beta}$ below are indicated by a label L . The harmonic gauge, where the coordinates at use satisfy $d * dx^\alpha = 0$, leads to the equation $\partial^\alpha h_{\alpha\beta} - \frac{1}{2}\partial_\beta h^\mu{}_\mu = 0$ and the linearized curvature forms in this gauge are

$$R^L_\alpha = -\square h_{\alpha\beta} dx^\beta, \quad R^L = -\square h^\alpha{}_\alpha \quad (16)$$

where $\square \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ for convenience [14]. From these expressions, it follows that the linearized Einstein 1-form is $G^L_\alpha = R^L_\alpha - \frac{1}{2}R^L dx^\alpha$ or equivalently $*G^L_\alpha = d * F^L_\alpha$. The contracted Bianchi identity $D * G^\alpha = 0$ holds at the linearized level in the form $d * G^L_\alpha = \partial_\beta G^{\alpha\beta}_L * 1 = 0$. With these preliminary considerations, the vacuum QC equations (7) can now easily be linearized around Minkowski background. (It is possible to linearize (7) around a constant curvature background which is also known to solve (7) exactly, see, e.g. [12]. Constant curvature backgrounds will not be considered in the discussion here) Since $*T^\alpha$ term involves only QC contributions, the linearization in this case involves only the constraint term $D\lambda^\alpha$, and writing out this term explicitly one finds

$$*E^\alpha = 2D * D[bR^\alpha + (2a + \frac{1}{2}b)R\theta^\alpha] + *T^\alpha = 0. \quad (17)$$

Therefore, after evaluating the exterior covariant derivatives in the constraint term and writing all the terms out explicitly, $*E^\alpha = 0$ then takes the form

$$\begin{aligned} & 2d * d[bR^\alpha + (2a + \frac{1}{2}b)R\theta^\alpha] + 2d * \{\omega^\alpha{}_\beta \wedge [bR^\beta + (2a + \frac{1}{2}b)R\theta^\beta]\} \\ & + 2\omega^\alpha{}_\beta \wedge *d[bR^\beta + (2a + \frac{1}{2}b)R\theta^\beta] + 2\omega^\alpha{}_\beta \wedge *\{\omega^\beta{}_\mu \wedge [bR^\mu + (2a + \frac{1}{2}b)R\theta^\mu]\} \\ & + \Omega^{\mu\nu} \wedge i^\alpha * [\theta_\mu \wedge (bR_\nu + aR\theta_\nu) - \theta_\nu \wedge (bR_\mu + aR\theta_\mu)] - i^\alpha \mathcal{L}_{qc}[a, b] = 0. \end{aligned} \quad (18)$$

DT charges were defined by expanding this equation around a constant curvature or flat background. In particular, the DT energy is obtained by linearizing the (00) component of the equation. As will explicitly be shown below, the definition of DT charge follows from the linearization of the first term in (18). Only the first term in (18) contains the terms linear in the partial derivatives of the metric components and it is remarkable that this particular term is an exact differential form in the full QC field equations. An important observation is that the linearization of the first term is obtained from $D\lambda^\alpha$ simply by replacing the covariant exterior derivatives with exterior derivatives and by adopting the linearized curvatures and the Hodge dual. Furthermore, by approximating the orthonormal coframe 1-forms as $\theta^\alpha \approx dx^\alpha$, the QC equations linearized in metric components then can be written as

$$*E^\alpha_L = d\lambda^\alpha_L = 2d * d[bR^\alpha_{L\beta} dx^\beta + (2a + \frac{1}{2}b)R_L dx^\alpha]. \quad (19)$$

In (19) and in all the linearized equations involving Hodge dual below, it is necessary that the Hodge dual is also to be approximated by that of the flat Minkowski space, for example, $\theta^\alpha \wedge * \theta^\beta \approx dx^\alpha \wedge * dx^\beta = \eta^{\alpha\beta} * 1$, etc. Now assuming that (19) is an expression in the flat background, then (19) explicitly yields

$$*E^\alpha_L = 2\partial_\nu \partial_\mu [bR^\alpha_{L\beta} + (2a + \frac{1}{2}b)\delta^\alpha_\beta R_L] dx^\nu \wedge *(dx^\mu \wedge dx^\beta). \quad (20)$$

By evaluating the inner products of the basis 1-forms indicated on the right hand side of (20) with the help of the identity [9]

$$dx^\nu \wedge *(dx^\mu \wedge dx^\beta) = -\eta^{\nu\mu} * dx^\beta + \eta^{\nu\beta} * dx^\mu \quad (21)$$

one eventually ends up with

$$*E^\alpha_L = 2P^{\alpha\mu} [bG^L_{\mu\beta} + (2a + b)\eta_{\mu\beta} R^L] * dx^\beta \quad (22)$$

where, following the notation in [3], the linear differential operator $P_{\alpha\beta}$ with respect to Minkowski background defined by

$$P_{\alpha\beta} \equiv \partial_\alpha \partial_\beta - \eta_{\alpha\beta} \square \quad (23)$$

is introduced for convenience. The crucial expression (23) for the projection operator $P_{\alpha\beta}$ results essentially from the contractions in the identity (21) which in turn follows from the general expression of λ^α in (9). In the above notation, $P_{\alpha\beta}$ can equally be defined by the action of (flat space) operator $d * d$ on a vector-valued 1-form $\sigma^\alpha = \sigma^\alpha_\beta dx^\beta$ by $d * d\sigma^\alpha = * \sigma'^\alpha$ with $\sigma'^\alpha \equiv P_{\mu\beta} \sigma^{\alpha\beta} dx^\mu$. It follows from the definition $P_{\alpha\beta}$ and linearized Bianchi identity $\partial_\alpha G^\alpha_{L\beta} = 0$ that $P_{\alpha\mu} G^\mu_{L\beta} = -\square G^\alpha_{L\beta}$ and therefore dropping the Hodge duals on both sides to write (22) component form, one obtains

$$E^L_{\alpha\beta} = -2[b\square G^\alpha_{L\beta} + (2a+b)(\eta_{\alpha\beta}\square - \partial_\alpha \partial_\beta)R^L]. \quad (24)$$

The expression in the square brackets is precisely the left hand side of Eqn. (5) in [3]. As it was shown in [3], one of the crucial properties of the (00) component of the expression is that it involves only the second order time derivatives of metric as a result of the fact that $P_{00} = -\nabla^2$ and consequently it corresponds to a Poisson type potential. Therefore the DT energy is related to the asymptotical behavior curvature rather than the asymptotical behavior of metric. Under certain assumptions about spatial asymptotics, (14) is therefore expected to yield results identical to DT charge by singling out leading terms of the curvature. Finding a DT charge for a given (asymptotically flat) solution of the QC equations requires the curvature to vanish as $\sim r^{-1}$ at spatial infinity and consequently, such solutions exclude the metrics having the asymptotic behavior $\sim r^{-1}$ [3]. Thus, one concludes that, without resorting to linear approximation, it is possible to define a conserved current for QC gravity and the DT energy definition corresponds to the linearized Lagrange multiplier term $d\lambda^0_L$ under certain assumptions about asymptotic behavior. Finally, note that the linearization of the current defined in (12) then corresponds to $T_{\mu\nu}$ side of Eqn. (5) in [3] up to a Hodge dual.

Now a brief discussion about the consistency of linearized equations and some properties of their solution is in order. Any solution of vacuum Einstein field equations is also a solution of the vacuum QC field equations (7), that is, $D\lambda^\alpha = 0$ and $*T^\alpha = 0$ separately because of $X^{\alpha\beta} = 0$ identically for a metric with $R^\alpha = 0$ in the above notation. However, the exact solutions of fourth order gravity has more integration constants [15] and even the linearized equations (and the solutions to them) have peculiar properties [16].

The property that the field equations are covariantly constant, $D * E^\alpha = 0$, continues to hold in the linear approximation. Explicitly, the linearized Bianchi identity reads

$$\partial_\alpha E^\alpha_{L\beta} = 2\partial_\alpha P^{\alpha\mu} [bG^\alpha_{L\mu} + (2a+b)\eta_{\mu\beta}R^L] \quad (25)$$

where $\partial_\alpha E^\alpha_{L\beta} = 0$ identically follows from the operator identity $\partial_\alpha P^{\alpha\mu} = 0$ by definition (23). In the presence of matter field sources, the field equations $*E_\alpha = -2 * T^\alpha_m$ coupled with matter energy-momentum 1-form $T^\alpha_m = T^\alpha_\beta \theta^\beta$ requires that at the linearized level, $\partial^\alpha T^\alpha_{m\beta} = 0$. The consistency of the linearized equations (22) require a further differential constraint on matter energy-momentum T^α_m (which is not present in the full non-linear equations). By calculating the trace of (22) and making use of the definition (23) one finds that linearization of $*E_\alpha = -2 * T^\alpha_m$ implies

$$\square T^\alpha_{m\beta} + \frac{2a+b}{2(3a+b)} P_{\alpha\beta} T^m = b\square^2 G^\alpha_{L\beta} \quad (26)$$

where T^m stands for the trace of the matter energy-momentum tensor. For the subcase $b = 0$, the constraint (26) was derived long ago [15] to show that linearized equations for R^2 gravity do not support solutions with the energy-momentum tensor of the form $T^\alpha_\beta = \delta^\alpha_0 \delta^\beta_0 \rho(\mathbf{x})$ for static, extended (and bounded) matter distribution $\rho(\mathbf{x})$. Explicitly, for the subcase $b = 0$, (00) component of (26) implies $\rho(\mathbf{x}) = 0$ [15]. For the general case $b \neq 0$ however, it yields $\rho(\mathbf{x}) \sim \Delta G^L_{00}$ where $P_{00} = \eta^{ij} \partial_i \partial_j \equiv \Delta$ and the Latin indices run over flat spatial coordinates. This is in accordance with the fact that the expression inside the square bracket in (22) can be regarded as a Poisson potential for the linear approximation of the theory [3].

IV. A CONSERVED CURRENT FOR NEW MASSIVE GRAVITY

The QC field equations (7) have the same form in any dimensions $n \geq 3$. For $a = n/4(n-1)$ and $b = 1$ in (5), the expression for the lagrange multiplier form in (9) becomes proportional to Cotton 2-form. In particular, for $n = 3$,

the lagrangian density for New massive gravity (NMG)[17] reads

$$\mathcal{L}_{NMG} = -R * 1 + \frac{1}{m^2} \mathcal{L}_K \quad (27)$$

up to an overall coupling constant which has been dropped for convenience. The parameter m has necessarily the dimension of mass and the QC part is $\mathcal{L}_K = \mathcal{L}_{qc}[a = -\frac{3}{8}, b = 1]$. At the linearized level (27) is equivalent to Pauli-Fierz lagrangian for a massive spin-2 particle.

In accordance with the above description, it is convenient to write the field equations for the QC part of (27) in the form given in (7) as $*E_K^\alpha = 0$. The vacuum equations that follow from (27) then take the form

$$*G^\alpha + \frac{1}{2m^2} *E_K^\alpha = 0. \quad (28)$$

Each of the terms in (28) can be split to define an exact pseudo-tensorial 2-forms, $*\tau_K^\alpha$ for the QC part and $*t^\alpha$ that is given in (3) for Einstein-Hilbert part. Consequently, (28) can be rewritten in a Maxwell-like form as

$$d * (F^\alpha + m^{-2} C^\alpha) + *\tau_{NMG}^\alpha = 0 \quad (29)$$

where $*\tau_{NMG}^\alpha$ explicitly reads

$$*\tau_{NMG}^\alpha = *t^\alpha + \frac{1}{m^2} (\omega^\alpha_\beta \wedge *C^\beta + \frac{1}{2} *T_K^\alpha) \quad (30)$$

by making use of the fact that $\lambda^\alpha = 2m^{-2} *C^\alpha$ with C^α standing for the Cotton 2-form. On the right hand side of (30), $*T_K^\alpha \equiv *T^\alpha[\Omega_{\mu\nu}, X_K^{\mu\nu}]$ with $X_K^{\mu\nu} \equiv X^{\mu\nu}[a = -\frac{3}{8}, b = 1]$. Consistently, at the linearized level, Eqn. (29) reads

$$d * (F_L^\alpha + m^{-2} C_L^\alpha) = 0 \quad (31)$$

(also assuming that at the linearized level $*$ stands for Minkowski Hodge dual) and (31) yields the field equation for a propagating massive spin-2 particle [17].

It is worth to emphasize that in (29) the $*F^0$ component of the Sparling form (the asymptotically dominant term) is related to ADM energy whereas $\lambda^0 = m^{-2} *C^0$ component is related to DT energy definition for generic higher curvature gravity. The new energy definition in Section II allows one to combine these energy definitions for NMG in a natural way.

As mentioned in the introduction, one of the motivations for the construction of the DT charges is to solve the zero energy problem of Weyl gravity with flat background. Weyl gravity follows from the lagrangian $\mathcal{L}_{qc}[a = -\frac{1}{3}, b = 1]$ in four dimensions and in this case, $\lambda^\alpha = 2 *C^\alpha$ as in NMG [12]. Consequently, in four dimensions P^α in (14) does not vanish identically in accordance with the DT result. On the other hand, for $n \geq 4$ dimensions, $\lambda^\alpha = 0$ identically for Gauss-Bonnet lagrangian density $\Omega_{\alpha\beta} \wedge \Omega_{\mu\nu} \wedge *\theta^{\alpha\beta\mu\nu}$ and the corresponding field equations are second order in metric components. Consequently, in this case P^0 in (14) yields zero identically for the energy as the DT definition of energy does as well.

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